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LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST.

[Continued from page 9.]

In demonstrating the important formulas (23) in the first part of this paper, we followed, as stated at p. 6, the analogy of a process employed by previous writers in finding a "multinomial formula" for the case of one variable. After the work was printed I perceived that for our present purpose it can be considerably shortened, without making it less intelligible.

Let us dispense with notation of a, B and n as in formulas (4) and (5), and designate the coefficients of the first power of the polynomial by λ as in (2), while those of the k power are represented by l. Then the given polynomial is

$$u=\begin{array}{l|l} \lambda_{-m,m}\ \xi^{-m}\eta^{m} & \lambda_{m,m}\ \xi^{m}\eta^{m} \\ \hline \lambda_{-m,-m}\xi^{-m}\eta^{-m} & \lambda_{m,-m}\xi^{m}\eta^{-m} \end{array},$$

and its expansion to the k power is denoted by

$$U=\begin{array}{l|l} l_{-km.km}\ \xi^{-km}\eta^{km} & l_{km,km}\ \xi^{km}\eta^{km} \\ \hline l_{-km.-km}\xi^{-km}\eta^{-km} & l_{km,-km}\xi^{km}\eta^{-km} \end{array}.$$

Differentiate these with respect to ξ, and in the first of the two equations (8),

$$kU\left(\frac{du}{d\xi}\right)=u\left(\frac{dU}{d\xi}\right),$$

substitute the expressions for u, U, $\frac{du}{d\xi}$ and $\frac{dU}{d\xi}$. This gives an equation similar to (10), each member of which contains the product of two rectangular polynomials. Forming the coefficient of $\xi^{i-1}\eta^{j}$ in each member, and equating the two to each other by the principle of indeterminate coefficients, we get

$$k\left(\begin{array}{l|l} -m\lambda_{-m,m}\ l_{(i+m),(j-m)} & m\lambda_{m,m}\ l_{(i-m),(j-m)} \\ \hline -m\lambda_{-m,-m}l_{(i+m),(j+m)} & m\lambda_{m,-m}l_{(i-m).(j+m)} \end{array}\right)$$
$$=\begin{array}{l|l} (i+m)\lambda_{-m.m}\ l_{(i+m),(j-m)} & (i-m)\lambda_{m,m}\ l_{(i-m),(j-m)} \\ \hline (i+m)\lambda_{-m.-m}l_{(i+m).(j+m)} & (i-m)\lambda_{m,-m}l_{(i-m),(j+m)} \end{array}. \qquad (24a)$$

Likewise, differentiating u and U with respect to η, and substituting in the second of the equations (8),

$$kU\left(\frac{du}{d\eta}\right)=u\left(\frac{dU}{d\eta}\right),$$

we get a second equation similar to (10), and equating to each other the coefficients of $\xi^{i}\eta^{j-1}$ in the product in each member, we have

$$k\left(\begin{array}{c|c} m\lambda_{-m,m}\ l_{(i+m),(j-m)} & m\lambda_{m,m}\ l_{(i-m),(j-m)} \\ \hline -m\lambda_{-m,-m}l_{(i+m),(j+m)} & -m\lambda_{m,-m}l_{(i-m),(j+m)} \end{array}\right)$$

$$=\begin{array}{c|c} (j-m)\lambda_{-m,m}\ l_{(i+m),(j-m)} & (j-m)\lambda_{m,m}\ l_{(i-m),(j-m)} \\ \hline (j+m)\lambda_{-m,-m}l_{(i+m),(j+m)} & (j+m)\lambda_{m,-m}l_{(i-m),(j+m)} \end{array}. \qquad (24b)$$

The relation existing between the coefficients λ and l which form the square groups (22), is expressed by these equations (24*a*) and (24*b*). Transposing in each of them, from the second member to the first, the portion which does not have the coefficient i or j, and rotating each table thro' 180° in its own plane, and changing the signs of both members, we get the desired formulas (23).

Now when the exponent k is made very large, or infinite, the coefficients l of the expansion become ordinates z to the limiting surface, and supposing them to be set close together so as to be consecutive, Δx and Δy are represented by dx and dy, and (24) reduces to

$$x = idx, \qquad y = jdy. \qquad (25)$$

The square group of coefficients l in (22) contains $(2m+1)^2$ terms, and the whole cxpansion contains $(2km+1)^2$ terms. Let k be made an infinity of the second order, as in the analogous case already cited (ANALYST, Sept. 1879, p. 147). The limiting surface will then extend over the whole of the infinite plane XY, and $(2km+1)^2$ will represent the whole number of points in that plane, or the whole number of consecutive ordinates z, while $(2m+1)^2$ is the number of such ordinates included in the square group under consideration; so that the portion of the plane XY occupied by this group is infinitesimal, both in its length and breadth. The corresponding portion of the surface may therefore be regarded as an element of a plane surface, that is, an element of the plane which is tangent to the limiting surface at the middle point of the group. Let z be restricted to mean the middle one of this square group of ordinates, answering to $l_{i,j}$ in (22), and let d_xz and d_yz denote the differentials of z taken in the x and y directions. Then the values of all the coefficients l in (22), at the limit, may be found from z by successive additions and subtractions of d_xz and d_yz, and will stand in the rectangular order

$$\begin{array}{c|c} z-md_xz+md_yz & z+md_xz+md_yz \\ \hline z-md_xz-md_yz & z+md_xz-md_yz \end{array}. \qquad (26)$$

Substituting them for the corresponding values of l in (23), collecting separately the coefficients of z, d_xz and d_yz in the result, remembering that

$$\begin{array}{c|c} \lambda_{-m,m} & \lambda_{m,m} \\ \hline \lambda_{-m,-m} & \lambda_{m,-m} \end{array} = 1 \qquad (27)$$

since the sum of all the given probabilities is certainty, also observing that we have identically

$$\left.\begin{aligned}
\frac{m(-m)\lambda_{m,-m} \;\big|\; m(m)\lambda_{-m,-m}}{-m(-m)\lambda_{m,m} \;\big|\; -m(m)\lambda_{-m,m}} &= \frac{-m(\;\;m)\lambda_{-m,m} \;\big|\; m(\;\;m)\lambda_{m,m}}{-m(-m)\lambda_{-m,-m} \;\big|\; m(-m)\lambda_{m,-m}} \\
\frac{-m(\;\;m)\lambda_{m,-m} \;\big|\; m(\;\;m)\lambda_{-m,-m}}{-m(-m)\lambda_{m,m} \;\big|\; m(-m)\lambda_{-m,m}} &= \text{ditto},
\end{aligned}\right\} \tag{28}$$

and using α_1, α_2, β_1, β_2 and γ as auxiliary letters, thus,

$$\left.\begin{aligned}
\alpha_1 &= \frac{-m\lambda_{-m,m} \;\big|\; m\lambda_{m,m}}{-m\lambda_{-m,-m} \;\big|\; m\lambda_{m,-m}}, & \alpha_2 &= \frac{m\lambda_{-m,m} \;\big|\; m\lambda_{m,m}}{-m\lambda_{-m,-m} \;\big|\; -m\lambda_{m,-m}}, \\
\beta_1 &= \frac{(-m)^2\lambda_{-m,m} \;\big|\; m^2\lambda_{m,m}}{(-m)^2\lambda_{-m,-m} \;\big|\; m^2\lambda_{m,-m}}, & \beta_2 &= \frac{m^2\lambda_{-m,m} \;\big|\; m^2\lambda_{m,m}}{(-m)^2\lambda_{-m,-m} \;\big|\; (-m)^2\lambda_{m,-m}}, \\
\gamma &= \frac{(-m)m\lambda_{-m,m} \;\big|\; mm\lambda_{m,m}}{(-m)(-m)\lambda_{-m,-m} \;\big|\; m(-m)\lambda_{m,-m}}, &&
\end{aligned}\right\} \tag{29}$$

and writing k instead of $k+1$, which is permissible because k is infinite, we shall find that (23) reduces to the form

$$\left.\begin{aligned}
k(-\alpha_1 z+\beta_1 d_x z+\gamma\, d_y z) &= -i(z-\alpha_1 d_x z-\alpha_2 d_y z), \\
k(-\alpha_2 z+\gamma\, d_x z+\beta_2 d_y z) &= -j(z-\alpha_1 d_x z-\alpha_2 d_y z).
\end{aligned}\right\} \tag{30}$$

These are differential equations of the limiting surface sought, and α, β, &c. are constant parameters whose significance we now inquire into. In all of them, the given probabilities λ occur in the same order as in (2) and (22), and regarding them as the masses of material points in a system, and taking the intervals between them, Δx and Δy, as units of distance, we see that here as in (3), α_1 and α_2 are the statical moments of the system about the Y and X axes through the middle of the group. If therefore the given elementary errors are so distributed about the point of no error, which is at the middle of the group, as to make this point the centre of gravity of the system of masses λ, the statical moments about any axes through this centre will be null, and we shall have $\alpha_1 = 0$ and $\alpha_2 = 0$. As for β_1 and β_2, the masses λ in each are multiplied by the squares of their distances from the Y and X axes respectively, so that β_1 and β_2 are the moments of inertia of the system about those , and they also represent the squares of the radii of gyration, since the sum of all the masses λ is unity. Lastly, γ is the sum of all the products formed by multiplying each λ into its distances from the Y and X axes.

It is proved in mechanics that if we have a system of masses M of material points in a plane, and draw any rectangular axes through their centre of gravity, and denote by u and v the coordinates of each point M, the "free axes" of the system in that plane, or the principal axes through the centre of gravity, will be determined in position by the equation

$$\tan 2\varphi = \frac{2\Sigma(Muv)}{\Sigma(Mu^2)-\Sigma(Mv^2)}, \tag{31}$$

where φ is the angle which a free axis makes with the assumed axis of the abscissas u. (See for instance the demonstration in Weisbach's *Mechanics.*) If the free axis coincides with this axis of abscissas, so that $\varphi = 0$, we shall have $\Sigma(Muv) = 0$. Now γ in (29) is formed in the same way as $\Sigma(Muv)$, whence it appears that if the given elementary errors are so distributed about the assumed X and Y axes through the point of no error, as to make these the free axes of the system of masses λ, we shall have not only $a_1 = 0$ and $a_2 = 0$, but $\gamma = 0$.

Of course, if the distribution of the elementary errors about the point of no error is entirely arbitrary, that point will not in general be the centre of gravity of their probabilities λ, and the assumed X and Y axes will not be free axes. If, however, we suppose these X and Y axes to be rotated about the origin, so as to become for instance parallel to the free axes, this will make no difference in the expansion. When the positions of the coefficients λ in a rectangular polynomial such as (2) or (22), and those of the coefficients l in its expansion, are referred to the same rectangular axes, these positions are independent of the directions of the axes so long as the origin remains the same. For example, let $x_1 y_1$ and $x_2 y_2$ be co-ordinates of two given points of error, referred to horizontal and vertical axes, and let $x'_1 y'_1$ and $x'_2 y'_2$ be co-ordinates of the same points referred to rectangular axes having the same origin but making an angle θ with the former. Multiplying together the two corresponding terms of the polynomial, omitting their coefficients λ, we have in the two systems

$$\left.\begin{aligned} \xi^{x_1}\eta^{y_1} \times \xi^{x_2}\eta^{y_2} &= \xi^{x_1+x_2}\eta^{y_1+y_2}, \\ \xi^{x'_1}\eta^{y'_1} \times \xi^{x'_2}\eta^{y'_2} &= \xi^{x'_1+x'_2}\eta^{y'_1+y'_2}. \end{aligned}\right\} \qquad (32)$$

The exponents in the second members are the coordinates of the place of the coefficient of the product, in the two systems. The known relations between such systems of coordinates as are here supposed, are

$$\left.\begin{aligned} x_1 &= x'_1\cos\theta - y'_1\sin\theta, & x_2 &= x'_2\cos\theta - y'_2\sin\theta, \\ y_1 &= x'_1\sin\theta + y'_1\cos\theta, & y_2 &= x'_2\sin\theta + y'_2\cos\theta, \end{aligned}\right\} \qquad (33)$$

and addition gives

$$\left.\begin{aligned} x_1 + x_2 &= (x'_1 + x'_2)\cos\theta - (y'_1 + y'_2)\sin\theta, \\ y_1 + y_2 &= (x'_1 + x'_2)\sin\theta + (y'_1 + y'_2)\cos\theta. \end{aligned}\right\} \qquad (34)$$

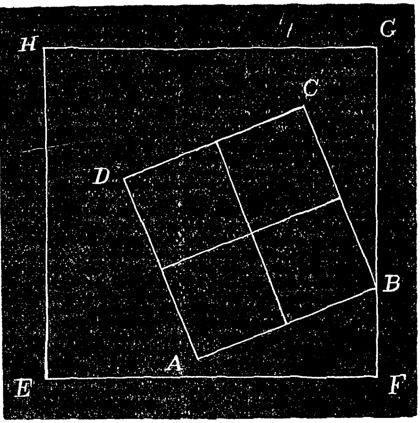
These equations, being of the same form as (33), show that just as $x_1 y_1$ and $x'_1 y'_1$ are coordinates of the same point in the two systems, so the point whose coordinates are $x_1 + x_2$ and $y_1 + y_2$ in the first system, is the same point whose coordinates in the other system are $x'_1 + x'_2$ and $y'_1 + y'_2$. In other words, the product occupies the same position under either system. And this being true for all the partial products which make up the total, it appears that whatsoever the directions of the assumed rectangular axes may be, the origin being unchanged, the places of the coefficients in the expanded polynomial will be the same, and consequently the limiting surface will be unchanged.

Supposing now that the assumed X and Y axes have been rotated so as to be parallel to the free axes, we may suppose further, that while the axes retain this new direction, the origin is transferred to some new point, for instance, to the centre of gravity of the masses λ. This also will make no difference in the positions relatively to each other, of the coefficients in the expansion to the k power. The only difference will be, that relatively to the origin, the whole system of coefficients in this expansion will be transferred through distances, in the x and y directions, k times as great as the distances through which the origin was transferred.

The change of origin amounts simply to increasing or diminishing all the exponents of ξ in (2) by one constant number, and the exponents of η by another, and this, as we have already noticed, does not alter the values of the coefficients in the expansion. If the expansions in the two systems are afterwards brought together so as to coincide, then their origins, or the places of exponent zero in each, will be separated by intervals, in the x and y directions, k times as great as the intervals between the origins in the first power. The case is similar to that of a polynomial of one variable, as treated in my former article (ANALYST, Sept. 1879, p. 146). It appears therefore, that no matter what point is taken as an origin, or what direction is given to the rectangular axes, the coefficients in the expansion of the given polynomial are always the same and in the same relative positions, and consequently their limiting surfaces are the same, differing only in position relatively to the axes. In any given case, we can choose the axes which are most convenient, and we shall choose the free axes of the system of masses λ, the origin being at the centre of gravity.

To show that our previous demonstration is applicable here, we observe that the units of measure Δx and Δy, which form the intervals between the positions of successive coefficients λ in the given polynomial (2), and correspond to differences of unity in the exponents of ξ and η, may be made as small as we please, or even infinitesimals, without affecting the positions of the coefficients, provided all the exponents up to m are made large in proportion. Fractional exponents can be converted into whole numbers by multiplying all the exponents of ξ by one sufficiently large number, infinite if necessary, and all those of η by another, and this will not alter the values of the coefficients in the expansion.

Our system of notation, by the way, represents the coefficient of any term in the first power of the polynomial by λ with sub-indices equal to the exponents which ξ and η may receive in that term. If now the origin, or place of $\xi^0\eta^0$, is transferred from one term to another, by adding or subtracting a constant number in all the exponents of ξ and another in those of η, this new origin can be made the middle of a square polynomial, by adding to the given terms a sufficient number of terms with zero coefficients, to complete the square. These terms are imagined merely to show that the conditions of the demonstration are satisfied, and their coeffiicients $\lambda=0$ disappear in the result (23).



Suppose for instance that the polynomial is such that its coefficients occupy the nine angular points of the four small squares into which the square *A B C D* is divided in the marginal figure, and that we wish to change the positions of the axes so that instead of being parallel to the sides of this square, with their origin at its centre, they shall have new directions and a new origin. Describe the square *EFGH* with its centre at the desired origin and its sides parallel to the new axes, and large enough to include within it the square *ABCD.* We may conceive that every point of the plane within this new square is occupied by a coefficient λ, only all of them are equal to zero except those at the nine angular points in the original sq. *ABCD.* All that our demonstration requires is that the coefficients λ in the given polynomial should form a square group with one term in the centre, the sides of the square being parallel to the axes of X and Y; and these conditions are fulfilled in the square *EFGH.* The number of terms which the square contains is $(2m + 1)^2$, and m here is of course infinite. But the relation (23) between these terms λ and the terms l in any square group of equal size in the expansion will still hold true, when the products λl are formed in the manner there stated. The first members of the two last eq's in (23) will represent the moments of the system of products λl, about axes passing through the centre of the square group and parallel to the new axes of X and Y. The numbers i and j will represent the coordinates of the centre of the square group of expanded terms l in the same X and Y system, the centre of the whole expansion being at the origin, so that i and j may be any whole numbers, up to infinity, and idx and jdy are the coordinates themselves. But a distance, or a lever arm, is not altered by changing the unit of measure. It makes no difference, for example, whether we say 3 feet, or 36 inches, or $(36\times\infty)$ times (1 inch $\div\ \infty$). We may restore then, if we please, in (22) and (23), the finite units of measure Δx and Δy, regarding m, i and j as finite numbers, only they will now, in general, be fractional. The same is true of the m in (26), for $d_x z$ and $d_y z$ are the increments of z corresponding to those increments in x and y respectively which are equal to the units of measure Δx and Δy, and which in the limiting surface are reprseented by dx and dy. Thus the validity of the differential eq's (30) of the limiting surface remains unimpaired by the change of axes, provided it is understood that the moments α, β, γ of the given system of masses λ, as expressed in (29), are now to be taken with reference to the new axes, that is, the free axes of the system, while the coordinates x and y, or idx and jdy, refer also to those axes.

Adopting therefore the centre of gravity of the coefficients λ as the origin, and their free axes as the axes of X and Y, we have in (29), as before stated, α_1, α_2 and γ equal to zero, and (30) takes the simple form

$$k\beta_1 d_x z = -iz, \qquad k\beta_2 d_y z = -jz. \tag{35}$$

Giving to i and j their values from (25) we can obtain

$$\frac{d_x z}{z} = \frac{-xdx}{k\beta_1 (dx)^2}, \qquad \frac{d_y z}{z} = \frac{-ydy}{k\beta_2 (dy)^2}. \tag{36}$$

According to what was said in connection with (29), $\beta_1 (dx)^2$ and $\beta_2 (dy)^2$ represent the squares of the radii of gyration of the masses λ about the X and Y axes, when Δx and Δy are represented by dx and dy at the limit. Let us write

$$r_1^2 = \beta_1 (dx)^2, \qquad r_2^2 = \beta_2 (dy)^2. \tag{37}$$

By the theorem which I gave in ANALYST, May 1880, p. 78, kr_1^2 and kr_2^2 represent the squares of the radii of gyration of the coefficients l in the whole expansion to the k power, so that $r_1\sqrt{k}$ and $r_2\sqrt{k}$ are the "quadratic mean errors" in the x and y directions. It is immaterial to the validity of that theorem, whether the radii of gyration are taken with reference to axes passing through the centre of gravity and parallel to the X and Y axes originally assumed, or are taken with reference to the free axes which stand at an angle φ with the others, for if the positions of the coefficients in the first power of the polynomial and its expansion are referred to any rectangular axes with origin at the centre of gravity, it will always hold true, as in my article cited, that the squares of the radii of gyration in the k power are k times the the squares of the corresponding radii in the first power. In that demonstration the exponents m, n, p and q may be as large as we please, even to infinity, and the units of measure Δx and Δy as small as we please, or infinitesimals. Whatever the direction of the axes may be, the polynomial can always be put in the form of a rectangle with sides parallel to the axes, by adding a sufficient number of terms having zero coefficients, as we have already noticed.

It is a known mechanical property of the free axes, that the radius of gyration is a maximum about one free axis and a minimum about the other. If then a given polynomial is raised to successive powers, making $k = 2$, $k = 3$, $k = 4$, &c., and the coefficients in these expansions are all referred to the same axes passing through the centre of gravity of the coefficients in the first power, the radii of gyration in the k power are $\sqrt{k}$ times what they are in the first power, for any given position of the axes, so that they are still a maximum when taken about one of the free axes of the first power, and a minimum abeut the other. Hence, the free axes retain a constant position, in all the successive powers.

If we now write

$$h_1^2 = \frac{1}{2kr_1^2}, \qquad h_2^2 = \frac{1}{2kr_2^2}, \tag{38}$$

the equations (36) become

$$\frac{d_x z}{z} = -2h_1^2 x\,dx, \qquad \frac{d_y z}{z} = -2h_2^2 y\,dy. \tag{39}$$

Their integral is

$$z = ce^{-(h_1{}^2x^2+h_2{}^2y^2)}. \tag{40}$$

They are readily derived from it by differentiating it with respect to x and y and dividing by z. To find the value of c, we consider that as the sum of all the coefficients λ in the first power of the polynomial is unity, the sum of all the coefficients z in its expansion to the k power must be unity. Hence arises the condition

$$\frac{1}{dx\,dy}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\,dx\,dy = 1, \tag{41}$$

which gives

$$\frac{c}{dx\,dy}\int_{-\infty}^{+\infty} e^{-h_1{}^2x^2}\,dx \int_{-\infty}^{+\infty} e^{-h_2{}^2y^2}\,dy = 1.$$

The known values of the two definite integrals are $\sqrt{\pi} \div h_1$ and $\sqrt{\pi} \div h_2$, so that we get

$$\frac{c}{dx\,dy}\left(\frac{\pi}{h_1 h_2}\right) = 1, \qquad \therefore c = \frac{h_1 h_2\,dx\,dy}{\pi}, \tag{42}$$

and the complete equation of the limiting surface stands

$$z = \frac{h_1 h_2\,dx\,dy}{\pi}\,e^{-(h_1{}^2x^2+h_2{}^2y^2)}. \tag{43}$$

[To be continued.]